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# Background Independence and the Open Topological String Wavefunction

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**Abstract**

The open topological string partition function in the background of a D-brane on a Calabi-Yau threefold specifies a state in the Hilbert space associated with the quantization of the underlying special geometry. This statement is a consequence of the extended holomorphic anomaly equation after an appropriate shift of the closed string variables, and can be viewed as the expression of background independence for the open-closed topological string. We also clarify various other aspects of the structure of the extended holomorphic anomaly equation. We conjecture that the collection of all D-branes furnishes a basis of the Hilbert space, and revisit the BPS interpretation of the open topological string wavefunction in this light.

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# 1 Introduction

In this paper, we study topological strings on Calabi-Yau threefolds with background D-branes. Our most basic motivation is to understand the properties of the open topological string amplitudes $\mathcal{F}^{(g,h)}$ that are implied by the extended holomorphic anomaly equations recently found in [1]. Two of the applications we have in mind are open-closed duality in the topological string, and the relation of the topological string to BPS state counting.

## 1.1 Background

Without attempting a complete history of the subject, we note that the holomorphic anomaly equations were originally obtained by BCOV [2, 3] as a kind of generalization of the $tt^*$-equations of Cecotti and Vafa [4] to higher-genus amplitudes. These equations arise as constraints on the amplitudes of certain topologically twisted $\mathcal{N} = 2$ superconformal field theories, and the coupling of the latter to topological gravity. The original $tt^*$-equations, which apply more generally also to topological twists of massive theories, have an appealing geometrical interpretation in the context of Frobenius manifolds [5], and have attracted much interest over the years. The mathematical scope of the holomorphic anomaly equations is somewhat less well understood, although of course when combined with mirror symmetry, they are a powerful tool to access the enumerative geometry of Calabi-Yau manifolds, see [6, 7] for the state of the art. Physically, the holomorphic anomaly equations have an elegant interpretation as the realization of "quantum background independence" of the topological string [8], also known as the "wavefunction interpretation". This slightly mysterious notion has recently played a central role in relating the topological string to BPS state counting [9].

One of the recurring themes in the literature on the $tt^*$-equations, the topological string, and the holomorphic anomaly equations is the relation to classical integrable systems and their quantization. Recent examples include the solution of the topological string on certain local Calabi-Yau manifolds in terms of matrix models [10], especially when viewed in the broader context of the duality between open and closed topological strings [11].

Our present work is concerned with some extensions of these structures to the situation with D-branes in the background. We believe that our results provide further hints of how the various topics listed above might be related at a deeper level. In par-

ticular, our results reinforce the point of view that D-branes and open-closed duality are central to a complete understanding of the wavefunction interpretation of the topological string, background independence, as well as the underlying integrable structure. Although there are notable differences from previous works, we attribute most of them to the distinction between local and compact setups. For example, the recent works [12, 13] have shown that topological string amplitudes that can be obtained from matrix models satisfy a set of equations essentially equivalent to the holomorphic anomaly equations.[1] In this situation, the Calabi-Yau geometry reduces to the spectral curve of the matrix model, and the resulting simplification in the Hodge structure should be the origin of the remaining discrepancies. Via open-closed duality for matrix models, we also anticipate a connection to the larger framework of [11], even though concrete attempts in this direction have so far been largely unsuccessful.

As another example, we mention that the wavefunction properties of the open topological string amplitudes on local Calabi-Yau manifolds have been previously discussed in [11, 16], see also [17]. In that context, the main discrepancies from our work seem to originate from the generic decoupling of open string moduli, or else the inaccessibility of certain D-brane charge sectors, on compact Calabi-Yau manifolds.

We hope that the clarification of the relation of our results with those and other works will help to isolate the central structures, and possibly even shed some light on the much more important problem of background independence in physical string theory, maybe along the lines of [18, 19, 20].

### 1.2 Results

In [21], it was noted that the general solution of the extended holomorphic anomaly equation of [1] can be mapped to a solution of the ordinary (BCOV) holomorphic anomaly equation by a certain shift of the closed string variables. This observation was then used to give a proof of the Feynman rule computation of the open topological string amplitudes given in [1], following [3]. Other recent work on the holomorphic anomaly for open strings includes [22, 23, 24], and see [25] for some earlier work.

Could such a simple relation between the open and closed topological string also hold for the *physical solutions* of the anomaly equations, namely, for the topological

[1] More precisely, the approach of [12, 13], which is based on new techniques in [14], will work quite generally for local Calabi-Yau manifolds which are conic bundles over complex curves, whether or not there is an underlying matrix model. See also [15] for recent progress in this direction.

amplitudes themselves? The holomorphic anomaly equations do not constrain the holomorphic dependence of the topological amplitudes, so there is clearly something non-trivial to check. On the other hand, the holomorphic part of the amplitudes is to a large extent constrained by a second set of equations, which express the statement that adding closed string insertions in worldsheet diagrams is equivalent to taking holomorphic (covariant) derivatives with respect to the moduli. In this way, one is actually reduced to the problem of determining, at each order in perturbation theory, a finite number of constants specifying the holomorphic part of the vacuum amplitudes. Fixing this "holomorphic ambiguity" normally requires additional information not contained in the holomorphic anomaly equation itself.

It turns out that when the open string amplitudes for non-trivial D-branes are shifted as in [21], they in fact *do not* obey the second, holomorphic, set of constraints. We hasten to emphasize that this does not affect the proof of the Feynman rules given in [21], because that proof only depends on the validity of the antiholomorphic constraints, namely the holomorphic anomaly equations. We can further mitigate the disappointment by revealing that there is a *different shift* of the open string amplitudes that leads to a solution of both the holomorphic and the antiholomorphic constraints.

We can give a brief summary of this result in formulas. In the simplest form, the BCOV holomorphic anomaly equations [3] are equivalent to a standard heat equation [26]:

$$\begin{aligned} \left[\frac{\partial}{\partial X^I} - \frac{1}{2} C_{IJK} \frac{\partial^2}{\partial y_J \partial y_K}\right] \Psi_{\text{closed}} &= 0\,, \\ \frac{\partial}{\partial \bar{X}^I} \Psi_{\text{closed}} &= 0\,, \end{aligned} \tag{1.1}$$

where $\Psi_{\text{closed}} := \Psi_{\text{closed}}(X^I, y_I)$, as a function of the $y_I$, is the generating function of the (closed) topological string amplitudes $\mathcal{F}^{(g)}_{i_1,\dots,i_n} := \mathcal{F}^{(g,0)}_{i_1,\dots,i_n}$, themselves functions of homogeneous coordinates $X^I$ on the closed string moduli space. In (1.1), $C_{IJK}$ is the three-point function on the sphere (Yukawa coupling), which is the basic data of the closed topological string at tree level.

As we will show, the extended holomorphic anomaly equations of [1] are equivalent to a heat equation extended by a "convection term,"

$$\begin{aligned} \left[\frac{\partial}{\partial X^I} - \frac{1}{2} C_{IJK} \frac{\partial^2}{\partial y_J \partial y_K} - i\mu\nu_{IJ}\frac{\partial}{\partial y_J}\right] \Psi_{\text{open}} &= 0\,, \\ \frac{\partial}{\partial \bar{X}^I} \Psi_{\text{open}} &= 0\,, \end{aligned} \tag{1.2}$$

for the generating function $\Psi_{\rm open}$ of the topological amplitudes $\mathcal{F}^{(g,h)}_{i_1,\ldots,i_n}$. Here $\nu_{IJ}$ is (part of) the disk amplitude with two bulk insertions, which is the basic holomorphic data specifying the D-brane background [1]. The tensor $\nu_{IJ}$ can be integrated to the superpotential or domain wall tension, $\nu_{IJ} = \partial_I \nu_J = \partial_I \partial_J \mathcal{T}$. We also introduced a formal real parameter $\mu$ counting the number of worldsheet boundaries.

It is fairly obvious that (1.2) can be transformed into (1.1) by a simple shift of the closed string variables, $y_I \to y_I - i\mu\nu_I$, as anticipated in [21].[2] In other words, given an open string background (specified by $\nu_{IJ}$), we define a *shifted* open string partition function by $\Psi^\nu(X^I, y_I) = \Psi_{\rm open}(X^I, y_I - i\mu\nu_I)$ (in this notation, $\Psi_{\rm closed} = \Psi^0$). This $\Psi^\nu$ then satisfies the ordinary heat equation (1.1), independent of $\nu_{IJ}$.

It is now meaningful to ask whether the shifted open string partition function is equal to the closed string partition function, in other words, whether $\Psi^\nu \overset{?}{=} \Psi^0$ is in fact independent of $\nu$. As it turns out, the answer is in the negative,[3] but it shines in a positive light when viewed instead as an answer to a long-standing question raised by Witten's interpretation of the holomorphic anomaly equation in the context of background independence, which we now recall.

### 1.3 A positive attitude

For fixed background $X^I$, Witten proposed [8] to view the topological string partition function $\Psi(X^I, y_I)$, as a function of the $y_I$, as a "wavefunction" specifying a quantum mechanical state in a particular presentation of a certain Hilbert space. Witten's Hilbert space, which we denote by $\mathcal{H}_W$, arises from the quantization of the symplectic vector space of topological ground states of the underlying worldsheet theory. The choice of background is equivalent to specifying a complex polarization of this vector space, and hence a particular presentation of the wavefunctions. The wavefunction depends on the background, but in a way that is completely fixed, according to the heat equation (1.1), by the variation of the polarization. The abstract quantum mechanical state itself, $|\Psi\rangle \in \mathcal{H}_W$, is independent of the background.

Our punchline might be clear already. A priori, the physical significance of Witten's auxiliary Hilbert space $\mathcal{H}_W$ is obscure if the closed topological string specifies only

[2] More precisely, the shift proposed in [21] would read in the present notation as $y_I \to y_I - i\mu(\nu_I - \bar\nu_I)$. We will discuss the difference between the two shifts extensively in section 3.1.

[3] Interestingly, the discrepancy between $\Psi^\nu$ and $\Psi^0$ arises *before* taking into account the holomorphic ambiguity.

one particular state in it. But if for any D-brane configuration whose topological amplitudes satisfy the extended holomorphic anomaly equation, the shifted partition function $\Psi^\nu$ satisfies the ordinary heat equation, this means precisely that any such D-brane specifies a state $|\Psi^\nu\rangle \in \mathcal{H}_W$. In fact, in section 4 we will describe evidence that, as $\nu$ varies over all possible D-brane configurations, the set of $|\Psi^\nu\rangle$ furnishes a basis of $\mathcal{H}_W$, thus filling Witten's entire Hilbert space with life.

It remains to be understood what physical principle selects the basis of states $|\Psi^\nu\rangle$, and in particular the closed string ground state $|\Psi^0\rangle$. To give some hints at the nature of this question, we note that when our topological string is the B-model on some Calabi-Yau threefold $Y$, then the basis $|\Psi^\nu\rangle$ is, at least partially, indexed by the set of all possible holomorphic curves in $Y$ (see section 4 for details). In other words, understanding the quantum Hilbert space of the topological B-model on $Y$ involves knowledge of all holomorphic curves in $Y$. Of course, the topological string knows a great deal about holomorphic curves on Calabi-Yau threefolds. Remarkably though, this knowledge arises from studying the A-model on $Y$, whereas we would here be trying to answer a B-model question (on the same manifold, not its mirror). This is suggestive of an intimate relation between topological A- and B-model on the same Calabi-Yau manifold, once D-branes are appropriately taken into account. This basic point has been emphasized by many people, beginning with [27], and is an ingredient in the topological S-duality proposal of [28, 29]. We also note that speculations along the above lines first appeared in the work of Donagi and Markman [30].

Let us now close this introduction and start with the derivation of the above-mentioned results. We will return to their interpretation in section 4, where we'll also include some more concrete speculations on the relation to BPS state counting.

## 2 Derivations

We are interested in the topological string obtained by twisting an $\mathcal{N} = 2$ superconformal field theory of central charge $\hat{c} = 3$ and all-integral $U(1)$ charges. As shown in [4, 3] in great generality, the space of chiral deformations of such a superconformal field theory carries the structure of a special Kähler manifold, which we will denote by $\mathcal{M}$, and which forms the basic holomorphic arena for the topological string. We will use local coordinates $t^i$, $i = 1, 2, \ldots, n = \dim_{\mathbb{C}}(\mathcal{M})$.

We note that our conventions in this section differ for convenience from those in

[1, 26], which will result in various factors of $i$ appearing differently.

### 2.1 Special geometry with D-branes

The central data of special geometry of $\mathcal{M}$ are the Hodge line bundle $\mathcal{L}$, with a Hermitian metric whose curvature is the special Kähler form of $\mathcal{M}$, and the Yukawa cubic $C$, which is a holomorphic section of $\mathcal{L}^{-2} \otimes \mathrm{Sym}^3 T^*\mathcal{M}$. The metric on $\mathcal{L}$ is denoted as $e^{-K}$ w.r.t. some local trivialization, providing a Kähler potential for the special Kähler metric on $\mathcal{M}$, $G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$. We will write $D$ generically for the metric-compatible connections on products of powers of $\mathcal{L}$ and $T$.

A crucial object is the "bundle of ground states" which we define as

$$V_{\mathbb{C}} := \mathcal{L} \oplus \mathcal{L} \otimes T\mathcal{M} \oplus \bar{\mathcal{L}} \otimes \bar{T}\mathcal{M} \oplus \bar{\mathcal{L}}\,, \tag{2.1}$$

where $T\mathcal{M}$ is the holomorphic tangent bundle of $\mathcal{M}$. $V_{\mathbb{C}}$ has an obvious conjugation operator $\bar{\cdot}$ which defines a real sub-bundle $V_{\mathbb{R}}$, a Hermitian metric $\langle \cdot | \cdot \rangle$ induced from those on $T\mathcal{M}$ and $\mathcal{L}$, and an antisymmetric bilinear form $\langle \cdot , \cdot \rangle$ defined by

$$\langle a, b \rangle := \langle \bar{a} | \sigma b \rangle \quad \text{where} \quad \sigma := \begin{cases} +i \text{ on } \mathcal{L} \oplus \bar{\mathcal{L}} \otimes \bar{T}\mathcal{M}\,, \\ -i \text{ on } \bar{\mathcal{L}} \oplus \mathcal{L} \otimes T\mathcal{M}\,. \end{cases} \tag{2.2}$$

We introduce an operator $\hat{C} : T\mathcal{M} \otimes V_{\mathbb{C}} \to V_{\mathbb{C}}$; for $v \in T\mathcal{M}$, $\hat{C}(v) \in \mathrm{End}(V_{\mathbb{C}})$ maps each space in (2.1) to its successor,

$$\hat{C}(v)|_{\mathcal{L}} = \cdot \otimes v\,, \quad \hat{C}(v)|_{\mathcal{L}\otimes T\mathcal{M}} = C(v) e^K G^{-1}\,, \quad \hat{C}(v)|_{\bar{\mathcal{L}}\otimes \bar{T}\mathcal{M}} = G(v, \cdot)\,, \quad \hat{C}(v)|_{\bar{\mathcal{L}}} = 0\,. \tag{2.3}$$

Similarly $\bar{\hat{C}}(v)$ maps each space in (2.1) to its predecessor. Using $\hat{C}$ we can define the "Gauss-Manin connection" on $V_{\mathbb{C}}$ in terms of its holomorphic and antiholomorphic parts,

$$\nabla = D - i\hat{C}\,, \quad \bar{\nabla} = \bar{D} + i\bar{\hat{C}}\,. \tag{2.4}$$

The connection preserves $V_{\mathbb{R}}$. Moreover, it is flat, as one verifies using the special geometry formula for the curvature of $G$. Hence it makes $V_{\mathbb{C}}$ into a holomorphic vector bundle, and there is a natural filtration by holomorphic subbundles

$$0 \subset F^3 V_{\mathbb{C}} \subset F^2 V_{\mathbb{C}} \subset F^1 V_{\mathbb{C}} \subset F^0 V_{\mathbb{C}} = V_{\mathbb{C}}\,, \tag{2.5}$$

where $F^k V_{\mathbb{C}}$ is the sum[4] of the first $4-k$ summands in (2.1).

[4] This is a direct sum of complex vector bundles; we emphasize however that its holomorphic structure induced from the Gauss-Manin connection is not a direct sum, because $\bar{\hat{C}}$ mixes the summands.

When the special Kähler manifold arises from twisting of an $\mathcal{N} = 2$ field theory, we have more data than what was mentioned above. The first extra datum is a twisted chiral ring, which in particular allows us to define the Euler characteristic,

$$\chi := 2n - 2\dim\big((a,c)\text{-ring}\big)\,. \tag{2.6}$$

The second extra datum is a lattice $V_{\mathbb{Z}}^* \subset V_{\mathbb{R}}^*$ preserved by the Gauss-Manin connection. We assume that $V_{\mathbb{Z}}^*$ is self-dual for the skew-symmetric pairing. Then we can choose an integral basis $\{\alpha^I, \beta_I\}$ for $V_{\mathbb{Z}}^*$, obeying

$$\langle \alpha^I, \alpha^J \rangle = 0\,, \quad \langle \alpha^I, \beta_J \rangle = \delta^I_J\,, \quad \langle \beta_I, \beta_J \rangle = 0\,. \tag{2.7}$$

Having fixed such a basis and a local section $\Omega$ of $\mathcal{L}$ one gets $2n+2$ functions on $\mathcal{M}$, the "periods"

$$X^I = \alpha^I(\Omega)\,, \quad F_I = \beta_I(\Omega)\,, \tag{2.8}$$

with $I = 0, \ldots, n$. The Kähler potential on $\mathcal{M}$ can then be written as

$$e^{-K} = i\big(X^I \bar{F}_I - \bar{X}^I F_I\big)\,. \tag{2.9}$$

Now we consider the open string sector. As argued in [31, 1], certain data of topological D-brane configurations on Calabi-Yau threefolds can be encoded holomorphically in the Hodge theoretic concept of *normal functions*. By definition, a normal function $\nu$ is a holomorphic section of the intermediate Jacobian fibration

$$J^3 := V_{\mathbb{C}}/(V_{\mathbb{Z}} + F^2 V_{\mathbb{C}}) \simeq (F^2 V_{\mathbb{C}})^* / V_{\mathbb{Z}}^* \tag{2.10}$$

satisfying Griffiths transversality,

$$\nabla \tilde{\nu} \in F^1 V_{\mathbb{C}}\,, \tag{2.11}$$

where $\tilde{\nu}$ is any lift of $\nu$ to $V_{\mathbb{C}}$, as well as certain growth conditions at infinities in $\mathcal{M}$ that will not be of central importance for our considerations. We shall not review here the discussion of the relation of D-branes with normal functions. We emphasize, however, that one of the fundamental ingredients in this identification is the decoupling of any continuous open string moduli from the topological string amplitudes. This might not apply in the most general situation, but we shall assume it in what follows. We also point out that the normal function need not capture the full information contained

in any given D-brane. To simplify our presentation, we use $\nu$ as a shorthand for the D-brane configuration.

The central point of [1] was the identification of the open string counterpart of the Yukawa cubic, which as reviewed above is the defining holomorphic data of the closed topological string at tree level. Recall that the Yukawa coupling is computed as the three-point function on the sphere. The open analogue is the two-point function on the disk, given geometrically by a particular non-holomorphic lift of the Griffiths infinitesimal invariant of the normal function $\nu$. A pedagogical reference on normal functions and their infinitesimal invariants is [32]. This lift is a section of $\mathcal{L}^{-1} \otimes \mathrm{Sym}^2 T^*\mathcal{M}$, and can be written as[5]

$$\Delta_{ij} = \langle \Omega, \nabla_i \nabla_j \tilde{\nu} \rangle = D_i D_j \mathcal{T} + i C_{ijk} e^K G^{k\bar{k}} D_{\bar{k}} \bar{\mathcal{T}} \,, \tag{2.12}$$

where $\mathcal{T}$ is a section of $\mathcal{L}^{-1}$ given by

$$\mathcal{T} = \langle \Omega, \nu \rangle \tag{2.13}$$

and we have chosen the unique *real* lift $\tilde{\nu} \in V_{\mathbb{R}}$.

Crucially, $\Delta_{ij}$ is not a holomorphic section. Instead, it satisfies a holomorphic anomaly equation,

$$\partial_{\bar{k}} \Delta_{ij} = i C_{ijk} \bar{\Delta}_{\bar{k}}^{k} \,, \tag{2.14}$$

where indices are raised with the metric, $\bar{\Delta}_{\bar{k}}^{k} = e^K G^{k\bar{j}} \bar{\Delta}_{\bar{k}\bar{j}}$.

A fundamental example of the structure described above is provided by Type II string theory on a Calabi-Yau threefold $Y$. In that case: $\mathcal{M}$ is the moduli space of complex structures on $Y$; $V_{\mathbb{C}} = H^3(Y, \mathbb{C})$ (similarly for $V_{\mathbb{R}}$, $V_{\mathbb{Z}}$); the decomposition (2.1) is the Hodge decomposition; and the Gauss-Manin connection is the standard flat structure provided by the deformation invariance of integer homology. The basic example of a normal function comes from a pair of homologically equivalent holomorphic curves $C_+$, $C_-$ in $Y$ varying over $\mathcal{M}$. Over every point in $\mathcal{M}$ we pick a three-cycle $\Gamma$ in $Y$ such that $\partial\Gamma = C_+ - C_-$. We obtain an element $\nu$ of $(F^2 V_{\mathbb{C}})^* \simeq (H^{3,0} \oplus H^{2,1})^*$ by associating to any $(3,0)$ or $(2,1)$-form $\omega$ the chain integral

$$\langle \omega, \nu \rangle = \int_\Gamma \omega \tag{2.15}$$

[5]This expression differs by a factor of $i$ from the corresponding expression in [1]. This factor can be traced back to a different convention for the Gauss-Manin connection, see (2.4).

viewed as a function over moduli space. The chain integral of cohomology classes in (2.15) is well-defined by Dolbeault's theorem and the holomorphicity of $\partial\Gamma$. Changing the choice of $\Gamma$ by a closed three-cycle, the integral changes by a period, which precisely accounts for the quotient by $V_{\mathbb{Z}}^*$ in (2.10).

The association (2.15) is known as the Abel-Jacobi map. It was first used in the context of mirror symmetry with D-branes on non-compact Calabi-Yau manifolds in [33, 34], and in a compact setting in [35, 31]. The relation (2.14) can be viewed as an open string analogue of special geometry [1]. See [36, 37] for a concurrent proposal.

### 2.2 Holomorphic anomaly of topological string amplitudes

We now fix some topological string background with D-branes characterized at tree-level by a normal function $\nu$ living over the special Kähler moduli space $\mathcal{M}$. We are interested in the perturbative topological string amplitudes $\mathcal{F}^{(g,h)}_{i_1,i_2,\ldots,i_n}$. These amplitudes are defined by integrating over the moduli space of Riemann surfaces of genus $g$, with $h$ boundary components, the appropriate correlators of the underlying worldsheet theory. The indices $i_1, i_2, \ldots, i_n$ stand for closed string insertions. See, *e.g.*, [1] for details.

As mentioned in the introduction, the $\mathcal{F}^{(g,h)}_{i_1,i_2,\ldots,i_n}$ satisfy two sets of iterative relations. The more obvious relations, which are holomorphic in nature, iteratively relate the amplitudes with insertions to those without. The amplitudes are non-zero for $6g + 3h + 2n - 6 \geq 0$, and we have in this case

$$\mathcal{F}^{(g,h)}_{i_1,i_2,\ldots,i_{n+1}} = D_{i_{n+1}} \mathcal{F}^{(g,h)}_{i_1,\ldots,i_n} \,. \tag{2.16}$$

The amplitudes with $6g + 3h + 2n - 6 < 0$ (all of which are tree-level amplitudes) are customarily set to zero, and we note that the vacuum amplitudes $\mathcal{F}^{(g,h)}$ for $2g+h-2 \geq 0$, the sphere three-point function $\mathcal{F}^{(0,0)}_{ijk} = C_{ijk}$, and the disk two-point function, $\mathcal{F}^{(0,1)}_{ij} = \Delta_{ij}$ are not constrained by (2.16) in any way.

The second set of relations are less obvious, but equally fundamental, and relate amplitudes on different worldsheet topologies,

$$\partial_{\bar{i}} \mathcal{F}^{(g,h)}_{i_1,\ldots,i_n} = \frac{1}{2} \sum_{\substack{g_1+g_2=g \\ h_1+h_2=h}} \bar{C}^{jk}_{\bar{i}} \sum_{s,\sigma\in S_n} \frac{1}{s!(n-s)!} \mathcal{F}^{(g_1,h_1)}_{j,i_{\sigma(1)},\ldots,i_{\sigma(s)}} \mathcal{F}^{(g_2,h_2)}_{k,i_{\sigma(s+1)},\ldots,i_{\sigma(n)}} +$$
$$\frac{1}{2} \bar{C}^{jk}_{\bar{i}} \mathcal{F}^{(g-1,h)}_{j,k,i_1,\ldots,i_n} + i\bar{\Delta}^{j}_{\bar{i}} \mathcal{F}^{(g,h-1)}_{j,i_1,\ldots,i_n} - (2g+h-2+n-1) \sum_{s=1}^{n} G_{i_s\bar{i}} \mathcal{F}^{(g,h)}_{i_1,\ldots,i_{s-1},i_{s+1},\ldots,i_n} \,. \tag{2.17}$$

These equations are valid for all $2g+h+n-2>0$, except for the one-point functions at one-loop $(g,h,n)=(1,0,1)$ or $(0,2,1)$, for which we have an additional term on the right hand side:

$$
\begin{aligned}
\partial_{\bar{i}}\mathcal{F}_j^{(1,0)} &= \frac{1}{2}C_{jkl}\bar{C}_{\bar{i}}^{kl} - \Big(\frac{\chi}{24}-1\Big)G_{j\bar{i}}\,,\\
\partial_{\bar{i}}\mathcal{F}_j^{(0,2)} &= i\Delta_{jk}\bar{\Delta}_{\bar{i}}^{k} - \frac{N}{2}G_{j\bar{i}}\,,
\end{aligned}
\tag{2.18}
$$

where $\chi$ is the Euler characteristic (2.6), and $N$ is the number of open string Ramond ground states of zero charge. The one-loop vacuum amplitudes are not constrained by (2.17) directly, but indirectly by (2.16) and (2.18). The holomorphicity of the sphere three-point function $\partial_{\bar{l}}C_{ijk}=0$, as well as the holomorphic anomaly of the disk two-point function, (2.14), appear as special cases of (2.17). (Here, one has to use the vanishing of the tree-level amplitudes with few insertions.)

The two sets of equations (2.16) and (2.17) together with their exceptional modification at one-loop and tree-level can be summarized more concisely in two "master equations" by introducing a certain generating function,

$$
\Psi(t^i,\bar{t}^{\bar{i}};x^i,\lambda^{-1}) = \lambda^{\frac{\chi}{24}-1-\mu^2\frac{N}{2}}\exp\Bigg[\sum_{\substack{g,h,n\\ 2g+h+n-2>0}}\frac{\lambda^{2g+h+n-2}}{n!}\mu^h\mathcal{F}^{(g,h)}_{i_1,\ldots,i_n}x^{i_1}\cdots x^{i_n}\Bigg]\,. \tag{2.19}
$$

With this definition, $\Psi$ is a section of the pullback of $\mathcal{L}^{\frac{\chi}{24}-1-\mu^2\frac{N}{2}}$ to the total space of $(\mathcal{L}\oplus\mathcal{L}\otimes T\mathcal{M})\to\mathcal{M}$. The variables $x^i$ are coordinates on $\mathcal{L}\otimes T\mathcal{M}$, the inverse string coupling $\lambda^{-1}$ is a coordinate on $\mathcal{L}$, and $\mu$ is a real parameter that keeps track of the number of boundaries in the expansion (2.19). $\Psi$ is holomorphic on each fiber but not holomorphic on the total space.

It is not hard to check that the relations (2.17) are equivalent to the following equation satisfied by the generating function $\Psi$:

$$
\Big[\partial_{\bar{i}} - \frac{1}{2}\bar{C}_{\bar{i}}^{jk}\frac{\partial^2}{\partial x^j\partial x^k} - G_{j\bar{i}}x^j\frac{\partial}{\partial\lambda^{-1}} - i\mu\bar{\Delta}_{\bar{i}}^{j}\frac{\partial}{\partial x^j}\Big]\Psi = 0\,. \tag{2.20}
$$

Also, the relations (2.16), together with the statements about the amplitudes with $2g+h+n-2\le 0$ (which are absent from (2.19)), are equivalent to

$$
\begin{aligned}
\Big[\partial_i - \Gamma_{ij}^k x^j\frac{\partial}{\partial x^k} + \partial_i K\Big(\lambda^{-1}\frac{\partial}{\partial\lambda^{-1}} + x^k\frac{\partial}{\partial x^k} + \frac{\chi}{24} - 1 - \mu^2\frac{N}{2}\Big)&\\
- \lambda^{-1}\frac{\partial}{\partial x^i} + \mathcal{F}_i^{(1,0)} + \mu^2\mathcal{F}_i^{(0,2)} + \frac{1}{2}C_{ijk}x^jx^k + \mu\Delta_{ij}x^j\Big]\Psi = 0\,.&
\end{aligned}
\tag{2.21}
$$

In the rest of this section, we will rewrite the equations (2.20) and (2.21) in various ways. The purpose is to show that the underlying structure is fairly simple, especially when viewed from the point of view of the so-called "large phase space". Readers interested primarily in the conceptual questions might be able to skip directly to section 3 on page 16.

## 2.3 The large phase space

To construct the large phase space [39], one first replaces the moduli space $\mathcal{M}$ by the total space $\widetilde{\mathcal{M}}$ of the line bundle $\mathcal{L}$ minus the zero section. One reason for introducing $\widetilde{\mathcal{M}}$ is that it comes equipped with very convenient coordinates, namely the $X^I$ we considered before. In other words, a choice of nowhere vanishing section of $\mathcal{L}$ defines an embedding of $\mathcal{M}$ into $\widetilde{\mathcal{M}}$ via

$$X^I = X^I(t^i)\,, \tag{2.22}$$

with inverse projection

$$t^i = t^i(X^I) \tag{2.23}$$

which is homogeneous of degree 0,

$$X^I \frac{\partial}{\partial X^I} t^i(X^I) = 0\,. \tag{2.24}$$

All quantities we consider are homogeneous of fixed degree under the overall rescaling of the $X^I$.

As complex vector bundles over $\mathcal{M}$, $\mathcal{L} \oplus \mathcal{L} \otimes T\mathcal{M} \simeq F^2 V_{\mathbb{C}}$; the pullback of $F^2 V_{\mathbb{C}}$ to $\widetilde{\mathcal{M}}$ is isomorphic to $T\widetilde{\mathcal{M}}$. The map from the pullback of $\mathcal{L} \oplus \mathcal{L} \otimes T\mathcal{M}$ to $T\widetilde{\mathcal{M}}$ is[6]

$$z^I = 2\,(\lambda^{-1} X^I + x^i X^I_{;i}) \tag{2.25}$$

where $z^I$ is the coordinate on the fiber of $T\widetilde{\mathcal{M}}$, and we defined

$$X^I_i := \partial_i X^I\,, \qquad X^I_{;i} := X^I_i + \partial_i K X^I\,. \tag{2.26}$$

Altogether, we have changed from the "small phase space" coordinates $(t^i, x^i, \lambda^{-1})$ on $(\mathcal{L} \oplus \mathcal{L} \otimes T\mathcal{M}) \to \mathcal{M}$ to the "large phase space" coordinates $(X^I, z^I)$ on $T\widetilde{\mathcal{M}}$.

[6] For convenience, this formula differs from that of [26] by an eighth root of unity.

Next we describe special geometry from the large phase space point of view, beginning with the closed string. The basic holomorphic data is encoded in the prepotential

$$F := \frac{1}{2} X^I F_I \tag{2.27}$$

and its first few derivatives,

$$F_I = \partial_I F\,, \qquad \tau_{IJ} := \partial_I \partial_J F\,, \qquad C_{IJK} := \partial_I \partial_J \partial_K F\,. \tag{2.28}$$

The homogeneity of $F$ implies a useful relation for the Yukawa coupling,

$$X^I C_{IJK} = 0\,. \tag{2.29}$$

Now we describe the large phase space version of the open string data, namely, normal functions and their infinitesimal invariants. The domain wall tension $\mathcal{T} = \langle \Omega, \nu \rangle$ is a period-like object, homogeneous of degree 1 in the large phase space:

$$\mathcal{T} = X^I \nu_I\,, \qquad \text{where } \nu_I := \partial_I \mathcal{T}\,. \tag{2.30}$$

The large phase space infinitesimal invariant is

$$\Delta_{IJ} := \langle \Omega, \partial_I \partial_J \nu \rangle = \nu_{IJ} - C^K_{IJ} \mathrm{Im} \nu_K\,, \tag{2.31}$$

where of course

$$\nu_{IJ} := \partial_I \partial_J \mathcal{T}\,, \qquad \text{with } X^I \nu_{IJ} = X^I \Delta_{IJ} = 0\,. \tag{2.32}$$

The last relation is very similar to (2.29) satisfied by the Yukawa coupling.

In Appendix A we give some useful relations between the large and small phase space data.

### 2.4 The anomaly equations in the large phase space

In this section we will rewrite the anomaly equations in the large phase space, which turns out to be the most convenient setting for understanding the relation between the open and closed anomaly equations.

We follow [38] and first solve the holomorphic anomaly equations for the one-loop amplitudes (2.18). Recall that the torus anomaly can be integrated to [2]

$$\mathcal{F}^{(1,0)} = -\frac{1}{2} \log \det \mathrm{Im} \tau_{IJ} - \left( \frac{\chi}{24} - 1 \right) K + f^{(1,0)} + \bar{f}^{(1,0)}\,, \tag{2.33}$$

where $f^{(1,0)}$ is a holomorphic ambiguity. The holomorphic anomaly equation of the annulus (as well as higher $(g,h)$) can be integrated by a procedure very similar to that in [3], see [1]. To this end, let us introduce the large phase space analogue, $\delta^J$, of the "terminator" of [1]. This quantity is defined by the equation

$$\bar{\Delta}_I^J = \bar{\partial}_I \delta^J \,, \tag{2.34}$$

which can be locally solved by

$$\delta^J = -2i \mathrm{Im}\tau^{JK} \mathrm{Im}\nu_K = \mathrm{Im}\tau^{JK}(\bar{\nu}_K - \nu_K) \,. \tag{2.35}$$

This choice of $\delta^J$ also satisfies

$$\partial_I \delta^J = -\Delta_I^J . \tag{2.36}$$

For future reference, we also record the holomorphic anomaly of the disk two-point function in large phase space:

$$\bar{\partial}_K \Delta_{IJ} = -\frac{i}{2} C_{IJL} \bar{\partial}_K \delta^L = -\frac{i}{2} C_{IJL} \bar{\Delta}_K^L \,. \tag{2.37}$$

With these definitions, the holomorphic anomaly equation of the annulus, which in the large phase space takes the form

$$\bar{\partial}_J \partial_I \mathcal{F}^{(0,2)} = -\frac{i}{2} \Delta_{IK} \bar{\Delta}_J^K + \frac{N}{2} \bar{\partial}_J \partial_I K \,, \tag{2.38}$$

can be integrated to

$$\begin{aligned} \mathcal{F}_I^{(0,2)} &= -\frac{i}{2} \Delta_{IK} \delta^K + \frac{1}{8} C_{IKL} \delta^K \delta^L + \frac{N}{2} \partial_I K + \partial_I f^{(0,2)} \,, \\ \mathcal{F}^{(0,2)} &= \frac{i}{4} \delta^K \mathrm{Im}\tau_{KL} \delta^L + \frac{N}{2} K + f^{(0,2)} + \bar{f}^{(0,2)} \,, \end{aligned} \tag{2.39}$$

where $f^{(0,2)}$ is another holomorphic ambiguity.

We absorb $f^{(1,0)}$ and $f^{(0,2)}$ into a redefinition of the generating function $\Psi$:

$$\Psi \rightarrow e^{-f^{(1,0)} - \mu^2 f^{(0,2)}} \Psi \,. \tag{2.40}$$

This should be interpreted as $\Psi_{\mathrm{old}} = e^{-f^{(1,0)} - \mu^2 f^{(0,2)}} \Psi_{\mathrm{new}}$; we will use this notation repeatedly in the next few sections.

It is then straightforward, using the formulas given in Appendix A, to show that the equations (2.20) and (2.21) become respectively

$$\begin{aligned} \Big[\bar{\partial}_I - \frac{1}{2} \bar{C}_I^{JK} \frac{\partial^2}{\partial z^J \partial z^K} + i\mu \bar{\Delta}_I^J \frac{\partial}{\partial z^J}\Big] \Psi = 0 \,, \\ \Big[\partial_I - \frac{1}{2} \partial_I \log\det \mathrm{Im}\tau_{IJ} + \frac{i}{2} C_{IJ}^K z^J \frac{\partial}{\partial z^K} + \frac{1}{8} C_{IJK} z^J z^K \\ + \frac{1}{2} \mu \Delta_{IJ} z^J - \frac{i}{2} \mu^2 \Delta_{IJ} \delta^J + \frac{1}{8} \mu^2 C_{IJK} \delta^J \delta^K \Big] \Psi = 0 \,. \end{aligned} \tag{2.41}$$

The closed string version of these equations was first obtained in [39]. Now note that by using

$$\Delta_{IJ} = \nu_{IJ} - C_{IJ}^{K}\mathrm{Im}\nu_K = \nu_{IJ} - \frac{i}{2}C_{IJK}\delta^K \tag{2.42}$$

we can rewrite the last four terms in (2.41) as

$$\begin{aligned}\frac{1}{8}C_{IJK}z^Jz^K + \frac{1}{2}\mu\nu_{IJ}z^J - \frac{i}{4}\mu C_{IJK}z^J\delta^K - \frac{i}{2}\mu^2\nu_{IJ}\delta^J - \frac{1}{8}\mu^2 C_{IJK}\delta^J\delta^K \\ = \frac{1}{8}C_{IJK}(z^J - i\mu\delta^J)(z^K - i\mu\delta^K) + \frac{1}{2}\mu\nu_{IJ}(z^J - i\mu\delta^J)\,.\end{aligned} \tag{2.43}$$

So if we now resubstitute

$$\Psi \to \sqrt{\det \mathrm{Im}\tau_{IJ}}\,\exp\Big[\frac{i}{4}(z^J - i\mu\delta^J)\mathrm{Im}\tau_{JK}(z^K - i\mu\delta^K)\Big]\Psi \tag{2.44}$$

and use the above relations for $\Delta_{IJ}$, $\delta^J$, etc., the equations take the form

$$\begin{aligned}\Big[\bar\partial_I - \frac{i}{2}\bar C_{IJ}^{K}z^J\frac{\partial}{\partial z^K} - \frac{1}{2}\bar C_I^{JK}\frac{\partial^2}{\partial z^J\partial z^K} + i\mu\bar\nu_I^J\frac{\partial}{\partial z^J}\Big]\Psi = 0\,, \\ \Big[\partial_I + \frac{i}{2}C_{IJ}^{K}z^J\frac{\partial}{\partial z^K}\Big]\Psi = 0\,.\end{aligned} \tag{2.45}$$

Finally, we change variables to $\bar y_I = \mathrm{Im}\tau_{IJ}z^J$ [26]. Geometrically this amounts to considering $\Psi$ as defined on $T^*\widetilde{\mathcal{M}}$ instead of $T\widetilde{\mathcal{M}}$. We then arrive at the simplest form of the anomaly equations,

$$\begin{aligned}\Big[\bar\partial_I - \frac{1}{2}\bar C_{IJK}\frac{\partial^2}{\partial\bar y_J\partial\bar y_K} + i\mu\bar\nu_{IJ}\frac{\partial}{\partial\bar y_J}\Big]\Psi = 0\,, \\ \partial_I\Psi = 0\,.\end{aligned} \tag{2.46}$$

For completeness, we summarize the sequence of redefinitions of $\Psi$ leading from (2.19) to (2.46):

$$\Psi_{(2.46)} = \frac{1}{\sqrt{\det \mathrm{Im}\tau_{IJ}}}\exp\Big[-\frac{i}{4}(z^J - i\mu\delta^J)\mathrm{Im}\tau_{JK}(z^K - i\mu\delta^K) + f^{(1,0)} + \mu^2 f^{(0,2)}\Big]\Psi_{(2.19)}\,. \tag{2.47}$$

After all these transformations $\Psi$ has turned out to be purely antiholomorphic. We can take $\Psi \to \overline{\Psi}$ to get

$$\begin{aligned}\Big[\partial_I - \frac{1}{2}C_{IJK}\frac{\partial^2}{\partial y_J\partial y_K} - i\mu\nu_{IJ}\frac{\partial}{\partial y_J}\Big]\Psi = 0\,, \\ \bar\partial_I\Psi = 0\,.\end{aligned} \tag{2.48}$$

To conclude this section we briefly discuss the global properties of $\Psi$. Before the redefinition $\Psi$ represented a section of the pullback of $\mathcal{L}^{\frac{\chi}{24}-1-\mu^2\frac{N}{2}}$, *i.e.* under a change of local section $\Omega \to e^f\Omega$ for $\mathcal{L}$ it transformed by $\Psi \to e^{(\frac{\chi}{24}-1-\mu^2\frac{N}{2})f}\Psi$. After the redefinition this transformation is canceled by the explicit transformations of $f^{(1,0)}$ and $f^{(0,2)}$ determined by (2.33) and (2.39). However, $f^{(1,0)}$, $f^{(0,2)}$, $z^J$, $\delta^J$ and $\tau$ appearing in (2.47) are all defined using a symplectic basis for $V_{\mathbb{Z}}$, and such a choice cannot be made globally on $\widetilde{\mathcal{M}}$ because $V_{\mathbb{Z}}$ can have global $Sp(2n+2,\mathbb{Z})$-valued monodromies; so the new $\Psi$ has to be considered as a section of a bundle which transforms appropriately under $Sp(2n+2,\mathbb{Z})$, *i.e.* as a modular form.

## 3 Discussion

The rewriting of the holomorphic anomaly equations in the last section makes the redefinition of variables which removes the open string data from the equation completely transparent. Shifting[7]

$$\bar{y}_I \to \bar{y}_I + i\mu\bar{\nu}_I \tag{3.1}$$

in (2.46) eliminates $\nu$ from both equations, and maps them onto the ordinary heat equations satisfied by the closed topological string amplitudes [26].

The existence of a shift removing open string data was first noted in [21]. With slightly different conventions, it was shown there that a general solution of the master holomorphic anomaly equation (2.20) can be mapped to a solution of the master equation with $\Delta_{ij} := 0$ (which is the master equation of BCOV) by shifting the closed string variables. The fate of the second equation (2.21) was however not analyzed in [21]. Instead, it was shown, using the above observation and the techniques of [3], that the solution of the perturbative holomorphic anomaly equations (2.17) can be written in a diagrammatic fashion using Feynman rules as noticed in [1].

In this section, we will first discuss the difference between the shift (3.1) and the shift of [21], at the level of the equation and at the level of the Feynman rules. We will then answer the question whether the simple shift maps open to closed topological string also at the level of the topological string amplitudes, which are the physically relevant solutions of the anomaly equations.

[7] Again, this shift should be interpreted as $(\bar{y}_I)_{\text{old}} = (\bar{y}_I)_{\text{new}} + i\mu\bar{\nu}_I$.

### 3.1 Shifts

The shift studied in [21] reads in the small phase space

$$x^i \to x^i - i\mu\bar{\Delta}^i\,, \qquad\qquad \lambda^{-1} \to \lambda^{-1} + i\mu\bar{\Delta}\,, \tag{3.2}$$

where $\bar{\Delta}^i$, $\bar{\Delta}$ are the terminators of [1] in the small phase space, satisfying

$$\bar{\partial}_{\bar{j}}\bar{\Delta}^i = \bar{\Delta}^i_{\bar{j}}\,, \qquad \bar{\partial}_{\bar{j}}\bar{\Delta} = G_{\bar{j}i}\bar{\Delta}^i\,, \tag{3.3}$$

where $\bar{\Delta}^i_{\bar{j}} = e^K G^{i\bar{i}}\bar{\Delta}_{\bar{j}\bar{i}}$.

To understand the relation between (3.1) and (3.2), we recall that the Griffiths infinitesimal invariant is the sum of two terms, which in small phase space are $\Delta_{ij} = \Delta^{(1)}_{ij} + \Delta^{(2)}_{ij}$, with $\Delta^{(1)}_{ij} = D_i D_j \mathcal{T}$ and $\Delta^{(2)}_{ij} = iC^{\bar{k}}_{ij} D_{\bar{k}}\bar{\mathcal{T}}$. In large phase space, the corresponding two terms are $\Delta_{IJ} = \Delta^{(1)}_{IJ} + \Delta^{(2)}_{IJ}$, with $\Delta^{(1)}_{IJ} = \nu_{IJ} + \frac{i}{2}C^K_{IJ}\nu_K$, $\Delta^{(2)}_{IJ} = -\frac{i}{2}C^K_{IJ}\bar{\nu}_K$. From this, we see that our shift (3.1) is by a potential for only the first term, $\bar{\partial}_I\bar{\nu}^J = (\bar{\Delta}^{(1)})^J_I$, whereas the shift (3.2) is by a potential for the full infinitesimal invariant, see (3.3).

In other words, the large phase space analogue of (3.2) would be

$$z^J \to z^J + i\mu\delta^J = z^J + i\mu(\bar{\nu}^J - \nu^J)\,. \tag{3.4}$$

It is not hard to see that under this shift, the open string data is eliminated only from the first equation in (2.41), whereas the second equation remains with an additional term, which in the large phase space reads

$$\left(-i\mu\nu^J_I\frac{\partial}{\partial z^J} - \frac{1}{2}\mu\nu_{IJ}z^J\right)\Psi\,. \tag{3.5}$$

Obviously, one may also write this term in the small phase space. It is worthwhile pointing out that the difference between the shift (3.1) and (3.2) can not be absorbed into a holomorphic ambiguity inherent in the definition of the terminators.

The proof of the Feynman rule expansion of open topological string amplitudes given in [21] relies on the same auxiliary finite-dimensional quantum system used in [3]. The dynamical variables are the $x = (x^i, \lambda^{-1})$. The quadratic part of the action is $Q(x,x)$, where $Q$ is an inverse to the collection of propagators $S^{ij}$, $S^i := S^{i\lambda^{-1}}$, $S := S^{\lambda^{-1}\lambda^{-1}}$. The interactions of the system are given by $\log\Psi(t^i,\bar{t}^{\bar{i}}; x^i - i\mu\bar{\Delta}^i, \lambda^{-1} + i\mu\bar{\Delta})$ where $\Psi$ is defined in (2.19) and the variables are shifted as in (3.2). This includes the vacuum amplitudes $\mathcal{F}^{(g,h)}$, and as non-trivial interaction vertices the infinite collection

of topological amplitudes $\mathcal{F}^{(g,h)}_{i_1,\ldots,i_n}$ with $n \geq 1$. As pointed out in [21], the shift (3.2) generates the possibility of terminating indices on $\bar{\Delta}^i$, $\bar{\Delta} =: \bar{\Delta}^{\lambda^{-1}}$, or in other words, it introduces a background vev for $(x^i, \lambda^{-1})$. This precisely reproduces the Feynman rules noticed in [1].

Our result is that to completely remove the open string data from both equations, one should shift only by (3.1) and absorb the second term, originating from $(\bar{\Delta}^{(2)})_I^J$, into the redefinition of $\Psi$ given in (2.47). This has a simple interpretation in terms of the finite-dimensional system discussed above. Namely, we give a smaller vev to $(x^i, \lambda^{-1})$ and instead introduce additional vertices $\sim D_i \mathcal{T} x^i$. The latter also behave like a tadpole, so that the overall Feynman rules are unaffected.

Referring to the perturbative holomorphic anomaly equations in (2.17), we notice that the second part of the infinitesimal invariant enters as if it were on equal footing with the closed string degenerations. Namely, it is of the form $(\bar{\Delta}^{(2)})_{\bar{i}}^{j} = -i\bar{C}_{\bar{i}}^{jk} D_k \mathcal{T}$, where $D_k \mathcal{T}$ could be viewed as a disk one-point function. The first part, $(\bar{\Delta}^{(1)})_{\bar{i}}^{j}$, however, can not be treated in this way and has to be accounted for in the master equation by a shift of variables (3.1).

We remark that the holomorphic anomaly equations for open topological strings derived in [13, 12] using matrix model duals to certain local Calabi-Yau manifolds appear to contain only the first type of contribution described in the previous paragraph, namely that originating from $\bar{\Delta}^{(2)}_{\bar{i}\bar{j}}$. It would be interesting to understand the Hodge theoretic origin of this simplification, as well as possible implications for open/closed duality in this context.

### 3.2 Solutions

We can now answer the question whether the simple shift of variables that maps the extended holomorphic anomaly equations to the ordinary BCOV equations will also transform correctly the actual topological string amplitudes. By studying the perturbative expansion in the small phase space, we will see that this possibility is in fact not realized.

The small phase space analogue of the shift (3.1) is

$$x^i \to x^i - i\mu\epsilon^i\,, \qquad \lambda^{-1} \to \lambda^{-1} + i\mu\epsilon\,, \tag{3.6}$$

where $\epsilon^i$, $\epsilon$ are potentials for the first part of the infinitesimal invariant. They are

defined by the equations

$$\begin{aligned}\partial_{\bar{i}}\epsilon^j &= e^K G^{j\bar{k}} D_{\bar{k}} D_{\bar{i}} \bar{\mathcal{T}}\,,\\ \partial_{\bar{i}}\epsilon &= G_{\bar{i}j}\epsilon^j\,,\end{aligned} \tag{3.7}$$

which can easily be solved by choosing

$$\begin{aligned}\epsilon &= e^K \bar{\mathcal{T}}\,,\\ \epsilon^i &= G^{i\bar{j}} e^K D_{\bar{j}} \bar{\mathcal{T}}\,.\end{aligned} \tag{3.8}$$

After the shift (3.6), the small phase space holomorphic anomaly equations (2.20) and (2.21) are transformed into

$$\begin{aligned}&\Big[\partial_{\bar{i}} - \frac{1}{2}\bar{C}_{\bar{i}}^{jk}\frac{\partial^2}{\partial x^j \partial x^k} - G_{j\bar{i}} x^j \frac{\partial}{\partial \lambda^{-1}} - \mu \bar{C}_{\bar{i}}^{jk} D_k \mathcal{T} \frac{\partial}{\partial x^j}\Big]\Psi = 0\,,\\ &\Big[\partial_i - \Gamma_{ij}^k x^j \frac{\partial}{\partial x^k} + \partial_i K\Big(\lambda^{-1}\frac{\partial}{\partial \lambda^{-1}} + x^k\frac{\partial}{\partial x^k} + \frac{\chi}{24} - 1\Big) - \lambda^{-1}\frac{\partial}{\partial x^i} + \mathcal{F}_i^{(1,0)} +\\ &\mu^2\Big(\mathcal{F}_i^{(0,2)} - \frac{N}{2}\partial_i K\Big) + \frac{1}{2}C_{ijk}(x^j - i\mu\epsilon^j)(x^k - i\mu\epsilon^k) + \mu\Delta_{ij}(x^j - i\mu\epsilon^j)\Big]\Psi = 0\,.\end{aligned} \tag{3.9}$$

The remaining $\mu$-dependent terms can be removed by the substitution

$$\Psi \to \exp\Big[-\mu D_k\mathcal{T} x^k - \mu\mathcal{T}\lambda^{-1} - \frac{\mu^2}{2} S^{jk} D_j\mathcal{T} D_k\mathcal{T} + \mu^2 S^k D_k\mathcal{T}\mathcal{T} - \mu^2 S\mathcal{T}^2 - \mu^2 f^{(0,2)}\Big]\Psi \tag{3.10}$$

where $S^{ij}$, $S^i$, and $S$ are the BCOV propagators, defined up to holomorphic ambiguity by the equations

$$\begin{aligned}\partial_{\bar{i}} S^{jk} &= \bar{C}_{\bar{i}}^{jk} = e^{2K} G^{j\bar{j}} G^{k\bar{k}} \bar{C}_{\bar{i}\bar{j}\bar{k}}\,,\\ \partial_{\bar{i}} S^j &= G_{\bar{i}k} S^{jk}\,,\\ \partial_{\bar{i}} S &= G_{\bar{i}k} S^k\,.\end{aligned} \tag{3.11}$$

To check that this substitution indeed removes the open string data, one has to use (among other things) the identity

$$\partial_{\bar{l}}\Big(\frac{1}{2}C_{ijk}\epsilon^j\epsilon^k + i\Delta_{ij}\epsilon^j + \frac{1}{2}\partial_i\big(S^{jk}D_j\mathcal{T}D_k\mathcal{T}\big) - \partial_i\big(S^j D_j\mathcal{T}\mathcal{T}\big) + \partial_i\big(S\mathcal{T}^2\big)\Big) = i\Delta_{ij}\bar{\Delta}_{\bar{l}}^j\,, \tag{3.12}$$

which provides the small phase space integration of the annulus anomaly (*cf.*, (2.39)).

Summarizing these transformations, the shifted open topological string partition function is

$$\begin{aligned}\Psi^\nu(t^i, \bar{t}^{\bar{i}}; x^i, \lambda^{-1}) = \exp\Big[&\mu D_k\mathcal{T} x^k + \mu\mathcal{T}\lambda^{-1} + \frac{\mu^2}{2} S^{jk} D_j\mathcal{T} D_k\mathcal{T}\\ &- \mu^2 S^k D_k\mathcal{T}\mathcal{T} + \mu^2 S\mathcal{T}^2 + \mu^2 f^{(0,2)}\Big]\Psi(t^i, \bar{t}^{\bar{i}}; x^i - i\mu\epsilon^i, \lambda^{-1} + i\mu\epsilon)\,,\end{aligned} \tag{3.13}$$

with $\Psi$ as defined in (2.19),

$$\Psi(t^i,\bar{t}^{\bar{i}};x^i,\lambda^{-1}) = \lambda^{\frac{\chi}{24}-1-\mu^2\frac{N}{2}} \exp\Bigg[\sum_{\substack{g,h,n\\2g+h+n-2>0}} \frac{\lambda^{2g+h+n-2}}{n!}\mu^h \mathcal{F}^{(g,h)}_{i_1,\ldots,i_n} x^{i_1}\cdots x^{i_n}\Bigg], \quad (3.14)$$

in terms of the open-closed topological string amplitudes with D-brane configuration given by the normal function, $\nu$. The statement is that all $\Psi^\nu$ satisfy the same holomorphic anomaly equations, identical to the anomaly equations satisfied by the closed topological partition function $\Psi^{\text{closed}} := \Psi^0$.

To see that the $\Psi^\nu$ are not all identical, it suffices to look at a few of the terms in the expansion of (3.13). For example, the leading behavior as $\lambda \to 0$ is given by $\lambda^{\frac{\chi}{24}-1-\mu^2\frac{N}{2}}$, and thus depends on $N$. More generally, because the sum in (3.14) is restricted to $2g+h+n-2>0$, the leading terms in the $\lambda$-expansion are given by the exponential prefactor in (3.13), and hence depend explicitly on the brane configuration.

In terms of the diagrammatic expansion discussed in the previous subsection, the fact that the shifted $\Psi^\nu$ all satisfy the same differential equation means that the Feynman rules, viewed as recursion relations between the $\mathcal{F}^{(g,h)}$, are independent of $\nu$. However, the different expansion around $\lambda = 0$ means that the initial conditions for the recursion relations are different. We stress that as a result the $\Psi^\nu$ are not determined by $\Psi^0$ alone, *even before taking into account the holomorphic ambiguity.*

# 4 Speculations

We have seen above that any D-brane configuration specified by a normal function $\nu$ determines a solution $\Psi^\nu$ of the ordinary holomorphic anomaly equation of BCOV. This is achieved by forming the generating function of open-closed topological string amplitudes, with a particular convention for disk and annulus amplitudes with few insertions, and then shifting the closed string variables in a certain way. This shift is different from the one proposed in [21]. We have explained this statement in various ways in both the small and large phase space. In this final section, we present three applications of our results.

## 4.1 Topological string wavefunctions

Witten has proposed in [8] that one should interpret the BCOV holomorphic anomaly equations as a statement of background independence in the topological string. Al-

though background independence does not hold order by order in perturbation theory, the all-genus partition function $\Psi(t^i, \bar{t}^{\bar{i}}; x^i, \lambda^{-1})$, as a function of $x^i$, $\lambda^{-1}$, can be thought of as a wavefunction defining a background independent state in a certain auxiliary "Hilbert space". Witten's Hilbert space, $\mathcal{H}_W$, arises by quantizing the symplectic vector space $V_{\mathbb{R}} = H^3(Y, \mathbb{R})$ in certain holomorphic polarizations, indexed by the choice of background $(t^i, \bar{t}^{\bar{i}})$. The holomorphic anomaly equations guarantee that as the background is varied, the wavefunction $\Psi(t^i, \bar{t}^{\bar{i}}; x^i, \lambda^{-1})$ varies precisely according to an infinitesimal Bogoliubov transformation, while the state $|\Psi\rangle \in \mathcal{H}_W$ is background independent.

In this interpretation of the holomorphic anomaly equation, the fact that the shifted open string partition functions $\Psi^\nu$ also satisfy the ordinary BCOV equation simply means that they also define states in the same Hilbert space,

$$|\Psi^\nu\rangle \in \mathcal{H}_W \qquad \text{for all } \nu\,. \tag{4.1}$$

The discussion in section 3 shows that $|\Psi^\nu\rangle$ indeed depends on $\nu$. We find it natural to conjecture that as $\nu$ varies over the set of all D-branes, the $|\Psi^\nu\rangle$ will furnish a basis of $\mathcal{H}_W$.

In support of this conjecture, we note that it is probably true at the semi-classical level, *i.e.*, at the level of the disk amplitude. Indeed, the topological disk partition function is given simply by the normal function itself. So we have to ask for the set of normal functions that can be realized by wrapping D-branes, where for concreteness we work in the context of the B-model on a Calabi-Yau $Y$. (Note that the usual definition of a normal function requires only Griffiths transversality, not that it be realizable algebraically.) It is by now well-accepted that the set of all B-branes on $Y$ is equivalent to the derived category of coherent sheaves, $D^b(Y)$. We obtain a normal function from any object in $D^b(Y)$ that deforms with $Y$ and whose second Chern class vanishes in $H^2(Y;\mathbb{Z})$ [40]. We gave the typical example of this at the end of subsection 2.1: Two homologous curves that deform with $Y$ define a normal function by integration over a bounding three-chain.

What is known mathematically is that the image of the so-called Griffiths group $\mathrm{Griff}^2(Y)$ (homologically trivial algebraic cycles of co-dimension two, modulo algebraic equivalence) under the Abel-Jacobi map to the intermediate Jacobian $J^3(Y)$ is not finitely generated as a vector space over $\mathbb{Q}$ [40, 41, 42]. It appears likely[8] that the

[8]We thank D. Morrison for conversations on this issue.

image of $\mathrm{Griff}^2(Y)$ in $J^3(Y)$ is in fact dense (in the analytic topology). Lifting the period ambiguity in the definition of a normal function, we conclude that the chain integrals (2.15) will also be dense in $V_{\mathbb{C}}/F^2 V_{\mathbb{C}} \simeq T^*\widetilde{\mathcal{M}}$.

Now Witten's Hilbert space $\mathcal{H}_W$ is a kind of quantization of the symplectic vector space $V_{\mathbb{R}}$, in a complex polarization which identifies it with a fiber of $T^*\widetilde{\mathcal{M}}$. Although the exact relation is somewhat cumbersome (particularly because the formal inner product in the complex polarization is not positive definite), we view the denseness of the image of the Abel-Jacobi map in $T^*\widetilde{\mathcal{M}}$ as evidence for the conjecture that the set of corresponding $|\Psi^\nu\rangle$ furnishes a ("Hilbert space") basis of $\mathcal{H}_W$.

### 4.2 BPS state counting

The wavefunction interpretation of the topological partition function plays a crucial role in the formulation of the OSV conjecture [9]. Namely, consider the Type IIB superstring compactified on $Y$. The resulting $d = 4$ supergravity theory includes electrically and magnetically charged BPS states, with charges $C \in V_{\mathbb{Z}}^*$. OSV conjectured that the degeneracies $\Omega(C)$ of these states are given by the Wigner function of $|\Psi\rangle$:

$$\Omega(C) = \langle\Psi|\mathcal{O}_C|\Psi\rangle\,, \tag{4.2}$$

where $\mathcal{O}_C$ is the Heisenberg group element associated to $C$. To write (4.2) more concretely one has to choose a polarization. For example, [9] uses the real polarization determined by a decomposition of the $V_{\mathbb{R}}$ into Lagrangian subspaces, $V_{\mathbb{R}} = V_{electric} \oplus V_{magnetic}$. Then $|\Psi\rangle$ is represented by a function $\Psi(\chi)$, $\chi \in V_{magnetic}$, the charge decomposes as $C = Q + P$, and (4.2) becomes

$$\Omega(Q,P) = \int d^n\chi\, \bar{\Psi}(\chi) \exp\left(Q^I \frac{\partial}{\partial\chi^I} + iP_I\chi^I\right) \Psi(\chi)\,. \tag{4.3}$$

As we have seen, each D-brane configuration $\nu$ provides another state $|\Psi^\nu\rangle$ in $\mathcal{H}_W$; so one might ask what its Wigner function computes. We are confused about various aspects of this question, but one possible guess follows. Recall from [1] that $\nu$ corresponds to a brane configuration with zero net charge. The simplest example is a pair of homologous holomorphic curves $E_\pm$ in $Y$. Consider wrapping a D3-brane of Type IIB on a 2-cycle in $Y$; this will give a string in the spatial $\mathbb{R}^3$, extended say in the $x^1$ direction. This string supports an effective field theory, with two supersymmetric vacua corresponding to wrapping the brane on the two curves $E_\pm$. Now consider a

"domain wall" configuration, i.e. Type IIB on $Y$ plus a brane which is in the $E_-$ vacuum as $x^1 \to -\infty$ and the $E_+$ vacuum as $x^1 \to \infty$. We regard this configuration as the "background". Possible BPS states in this background include particles in the bulk of $\mathbb{R}^3$ as well as ones bound to the domain wall on the string. So a minimally invasive modification of the OSV conjecture is to propose that the Wigner function of $|\Psi^\nu\rangle$ counts such states. (The quantization of charge in this sector is non-standard — there is a fractional part determined by $\nu$. We do not understand how this affects the proposal.)

An open string extension of the OSV conjecture was also proposed in [16]. In that case, however, one considers branes in a non-compact Calabi-Yau. This leads to several salient differences from the compact case: the topological partition function depends on the continuous moduli of the brane, and the physical theory includes additional BPS states which couple to them. As a result the proposal there took a somewhat different form.

### 4.3 Hints from supersymmetry

We have found that the holomorphic anomaly equations of the open and closed topological string can be transformed into one another by a simple shift of variables, similar to that given in [21]. This simple statement deserves to have a physical interpretation. In searching for one, however, we encounter an immediate puzzle: the variables we shifted were not the physical moduli $(t^i, \bar{t}^{\bar{i}})$ of the worldsheet theory, but rather the formal generating-function parameters $(x^i, \lambda^{-1})$. So to find a physical explanation of the shift we have to find a physical meaning for these parameters.

A possible clue comes from the fact that the equations and the shift take their simplest form when we trade the coordinates $(t^i, \bar{t}^{\bar{i}}, x^i, \lambda^{-1})$ for $(X^I, y_I)$, *i.e.* we use the complex structure of $T^*\widetilde{\mathcal{M}}$, as was done in [26]. This complex manifold has a natural meaning from the spacetime point of view. In the superconformal approach to $\mathcal{N} = 2$ supergravity coupled to the vector multiplet moduli space $\mathcal{M}$, one begins by considering a rigid (non-gravitational) theory with vector multiplet moduli space $\widetilde{\mathcal{M}}$. Upon classical dimensional reduction of this rigid theory from $d = 4$ to $d = 3$, say along $x^3$, supersymmetry dictates that one obtains a theory with a hyperkähler moduli space. This space turns out to be (in one of its complex structures) exactly $T^*\widetilde{\mathcal{M}}$ [43]. So from this point of view the formal parameters $y_I$ become physical: they represent the electric and magnetic Wilson lines $(A_I)_3, (A_I^D)_3$ of the $d = 4$ gauge fields along the

$x^3$ direction. It would be very interesting to understand whether the shift relating the open and closed string can be related to turning on these gauge fields. As support for this idea note that $\lambda$ certainly is related to the graviphoton field strength [3].

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## A From large to small phase space

In this appendix we give a few formulas which are needed for passing between the large and small phase space.

Recall the definition (2.26),

$$X^I_i := \partial_i X^I, \qquad X^I_{;i} := X^I_i + \partial_i K X^I. \tag{A.1}$$

Tangent directions to $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are related by

$$\frac{\partial t^j}{\partial X^J} X^J_i = \delta^j_i\,, \quad \frac{\partial t^i}{\partial X^J} X^I_{;i} = \delta^I_J - e^K \bar{X}^L 2\mathrm{Im}\tau_{LJ} X^I. \tag{A.2}$$

The natural metric on $\widetilde{\mathcal{M}}$ is $\mathrm{Im}\tau$, related to the special Kähler data $(G,e^K)$ on $\mathcal{M}$ by

$$e^{-K} = 2\bar{X}^I X^J \mathrm{Im}\tau_{IJ}\,, \qquad e^{-K} G_{i\bar{j}} = -X^I_{;i}\bar{X}^J_{;\bar{j}}\, 2\mathrm{Im}\tau_{IJ}\,, \qquad \bar{X}^M 2\mathrm{Im}\tau_{MI} X^I_{;j} = 0\,. \tag{A.3}$$

The inverse metrics are similarly related by

$$e^K G^{i\bar{j}} = -\frac{\partial t^i}{\partial X^I}\frac{\partial \bar{t}^{\bar{j}}}{\partial \bar{X}^J}\,\frac{1}{2}\mathrm{Im}\tau^{IJ}\,, \qquad X^I_{;i}\bar{X}^J_{;\bar{j}} e^K G^{i\bar{j}} = -\frac{1}{2}\mathrm{Im}\tau^{IJ} + e^K X^I \bar{X}^J\,. \tag{A.4}$$

The connection and Yukawa coupling on $\mathcal{M}$ are related to the data on $\widetilde{\mathcal{M}}$ by

$$\Gamma^k_{ij} - \partial_i K \delta^k_j = \frac{\partial t^k}{\partial X^I}\partial_i(X^I_{;j}) - \frac{i}{2}\frac{\partial t^k}{\partial X^I} X^M_{;i} X^K_{;j} C^I_{KM}\,, \tag{A.5}$$

$$C_{ijk} = X^I_i X^J_j X^K_k C_{IJK} = X^I_{;i} X^J_{;j} X^K_{;k} C_{IJK}\,. \tag{A.6}$$

where we have used the homogeneity (2.29) of the Yukawa coupling. Using the foregoing relations one can verify in particular that $\Gamma^k_{ij}$ and $C_{ijk}$ have the expected transformation properties.

Another useful fact, which may be verified using the last equation of (A.3), is

$$e^K 2\bar{X}^M 2\mathrm{Im}\tau_{MI}\partial_i(X^I_{;j})x^j = \frac{i}{2}X^M_i C^I_{KM} e^K \bar{X}^N 2\mathrm{Im}\tau_{NI} z^K \,. \tag{A.7}$$

Finally, the infinitesimal invariants as defined in (2.12) and (2.31) are related by

$$\Delta_{ij} = X^I_i X^J_j \Delta_{IJ} = X^I_{;i} X^J_{;j} \Delta_{IJ} \,. \tag{A.8}$$

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